

# UNIQUE COLORING OF COMPLETE GRAPHS REMOVE SOME DISJOINT EDGES

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## ABSTRACT

A graph is called chromatically unique if it is isomorphic to every graph that has the same coloring polynomial. Given a complete graph  $K_n$  with vertex set  $V(K_n)$  and edge set  $E(K_n)$ . Consider the graph  $G^* = (V^*, E^*)$  with  $V^* = V(K_n)$  and  $E^* = E(K_n) \setminus \{e_1, e_2, \dots, e_m\}$ , where  $e_1, e_2, \dots, e_m$  are disjoint edges in  $E(K_n)$ . A graph  $G^*$  is called a complete graph remove some disjoint edges. In this paper, we will prove that the graph  $G^*$  is chromatically unique.

**Keywords:** Chromatic number, coloring polynomial, coloring equivalence, unique coloring.

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## 1. INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If  $G$  is a graph, then  $V(G)$  and  $E(G)$  (or  $V$  and  $E$  in short) will denote its vertex-set and its edge-set, respectively. The subgraph of  $G$  induced by  $W \subseteq V(G)$  is denoted by  $G[W]$ . A graph with  $n$  vertices and no edges is called an *empty graph*, denoted by  $O_n$ . A graph with  $n$  vertices and any two distinct vertices are adjacent is called a *complete graph*, denoted by  $K_n$ . A graph  $G = (V, E)$  is called *k-partite graph* if there exists a partition of  $V$  into  $k$  classes  $V = V_1 \cup V_2 \cup \dots \cup V_k$  such that the subgraphs of  $G$  induced by  $V_i, i = 1, 2, \dots, k$ , is independent set, denoted by  $G = (V_1 \cup V_2 \cup \dots \cup V_k, E)$ . An *r-partite graph* in which every two vertices from different partition classes are adjacent is called complete *r-partite graph* and is

denoted by  $K_{|V_1|, |V_2|, \dots, |V_k|}$ . Given graph  $G = (V, E)$  and  $U \subseteq V$ , if we delete all vertices belonging to  $U$  along with all edges associated with them, we will get a graph denoted as  $G-U$ . Two graphs  $G$  and  $H$  are said to be *isomorphic* to each other if there exists a map  $f: V(G) \rightarrow V(H)$  such that  $uv \in E(G)$  if and only if  $f(u)f(v) \in E(H)$  with  $u, v \in V(G)$ , denoted by  $G \cong H$ . Given graphs  $G$  and  $H$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$  then we say graph  $H$  is a subgraph of graph  $G$ , denoted by  $H \subseteq G$ . Unless otherwise indicated, our graph-theoretic terminology will follow [1].

Let  $G = (V, E)$  be a graph and  $\lambda$  is a positive integer. A  $\lambda$ -coloring of  $G$  is a mapping  $f: V(G) \rightarrow \{1, 2, \dots, \lambda\}$  such that  $f(u) \neq f(v)$  for any adjacent vertices  $u, v \in V(G)$ . The smallest positive integer  $\lambda$  such that  $G$  has a  $\lambda$ -coloring is called the *chromatic number* of  $G$  and is denoted by  $\chi(G)$ . We say that a graph  $G$  is *n-chromatic* if  $n = \chi(G)$ . It is easy to see that  $\chi(O_n) = 1$  and  $\chi(K_n) = n$ .

Let  $V(G) = \{v_1, v_2, \dots, v_n\}$ , two  $\lambda$ -colorings  $f$  and  $g$  are considered different if and only if  $f(v_k) \neq g(v_k)$  for some  $k = 1, 2, \dots, n$ . Let  $P(G, \lambda)$  (or simply  $P(G)$ ) denote the number of distinct  $\lambda$ -colorings of  $G$ . It is well-known that for any graph  $G$ ,  $P(G, \lambda)$  is a polynomial in  $\lambda$ , called the *chromatic polynomial* of  $G$ . The notion of chromatic polynomials was first introduced by Birkhoff [2] in 1912 as a quantitative approach to tackle the four-color problem. It is easy to see that the coloring polynomial of the graph  $G = O_n$  is  $P(G, \lambda) = \lambda^n$  và the coloring polynomial of the graph  $G = K_n$  is  $P(G, \lambda) = \lambda(\lambda-1)\dots(\lambda-n+1)$ . Two graphs  $G$  and  $H$  are called *chromatically equivalent* or in short  $\chi$ -equivalent, and we write in notation  $G \sim H$ , if

$P(G, \lambda) = P(H, \lambda)$ . A graph  $G$  is called *chromatically unique* or in short  $\chi$ -unique if  $G \cong G'$  for any graph  $G'$  such that  $G \sim G'$ . The notion of  $\chi$ -unique graphs was first introduced and studied by Chao and Whitehead [3] in 1978. It is easy to see that the empty graph  $O_n$  and the complete graph  $K_n$  are the chromatically unique graphs. To learn more about chromatic polynomials, chromatically equivalent graphs and chromatically unique graphs, readers should study more references [4-6]. There are also some recent research results on this topic, presented in the references [7-10].

In this paper, we continue to study the coloring problem for the class of complete graphs remove some disjoint edges. We determine the chromatic number and prove that this class of graphs is chromatically unique.

## 2. MATERIALS AND METHODS

### 2.1. Complete graphs remove some disjoint edges and its chromatic number

Given the complete graph  $K_n$  with vertex set  $V(K_n) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(K_n)$ . Suppose that  $e_1, e_2, \dots, e_m$  are disjoint edges in  $E(K_n)$  with  $2 \leq 2m \leq n$ . We consider the graph  $G^* = (V^*, E^*)$  as follows

$$V^* = V(K_n) = \{v_1, v_2, \dots, v_n\},$$

$$E^* = E(K_n) \setminus \{e_1, e_2, \dots, e_m\}.$$

A graph  $G^*$  is called a *complete graph remove some disjoint edges*. Without loss of generality, we can assume that  $e_1 = v_1v_2, e_2 = v_3v_4, \dots, e_m = v_{2m-1}v_{2m}$ . Put

$$V_1^* = \{v_1, v_2\}, V_2^* = \{v_3, v_4\}, \dots,$$

$$V_m^* = \{v_{2m-1}, v_{2m}\}, V_{m+1}^* = \{v_{2m+1}\}, \dots, V_{n-m}^* = \{v_n\}$$

First, we determine the number of edges of the graph  $G^*$ .

**Lemma 1.** *The number of edges of the graph  $G^* = (V^*, E^*)$  is*

$$|E(G^*)| = \frac{n^2 - n - 2m}{2}$$

*Proof.* It is easy to see that graph  $G^*$  is the graph  $K_{t_1, t_2, \dots, t_m, t_{m+1}, \dots, t_{n-m}}$  with

$$t_1 = t_2 = \dots = t_m = 2, t_{m+1} = \dots = t_{n-m} = 1$$

We have

$$|E(G^*)| = \sum_{1 \leq t_i < t_j \leq n-m} t_i t_j$$

$$= 2^2 \cdot (1 + 2 + \dots + m - 1) + 2m(n - 2m) + (1 + 2 + \dots + n - 2m - 1)$$

$$= 2m(m - 1) + 2m(n - 2m) + \frac{(n - 2m - 1)(n - 2m)}{2}$$

$$= \frac{n^2 - n - 2m}{2}$$

Next, we determine the relationship between the chromatic number of the graph  $G$  and the chromatic number of the graph  $H$  when  $H \subseteq G$ .

**Lemma 1** ([1]). *If  $H \subseteq G$  then  $\chi(H) \leq \chi(G)$ .*

Next, we recall the chromatic number of the complete graph by the following lemma.

**Lemma 2** ([1]). *If  $K_n$  is a complete graph on  $n$  vertices then  $\chi(K_n) = n$ .*

Now we will determine the chromatic number of the graph  $G^* = (V^*, E^*)$ .

**Lemma 3.**  $\chi(G^*) = n - m$ .

*Proof.* It is easy to see that the graph  $G^*$  contains a complete subgraph of order  $n - m$   $H = (V(H), E(H))$  with  $V(H) = \{v_1, v_3, \dots, v_{2m-1}, v_{2m+1}, \dots, v_n\}$ . By Lemmas 1 and 2,  $n - m = \chi(H) \leq \chi(G^*)$ . Next we will prove that using colors  $1, 2, \dots, n - m$  we can color the graph  $G^*$ . Indeed, suppose  $f: V(G^*) \rightarrow \{1, 2, \dots, n - m\}$  is a coloring of  $G^*$  as follows:

$$f(v_1) = f(v_2) = 1, f(v_3) = f(v_4) = 2, \dots,$$

$$f(v_{2m-1}) = f(v_{2m}) = m, f(v_{2m+1}) = m + 1, \dots, f(v_n) = n - m$$

Obviously,  $f$  is  $(n - m)$ -coloring of  $G^*$ . Thus,  $\chi(G^*) = n - m$ .

### 2.2. Some properties of two chromatically equivalent graphs

The following lemma will tell us the relationship between the number of vertices, the number of edges, the chromatic number and the connectedness of two chromatically equivalent graphs.

**Lemma 4** ([3]). *Let  $G$  and  $H$  be two  $\chi$ -equivalent graphs. Then*

- (i)  $V(G) = V(H)$ ;
- (ii)  $E(G) = E(H)$ ;

- (iii)  $\chi(G) = \chi(H)$ ;
- (iv)  $G$  is connected if and only if  $H$  is connected;
- (v)  $G$  is 2-connected if and only if  $H$  is 2-connected.

### 3. RESULTS AND DISCUSSION

First we state and prove the following lemma. This is an important lemma that helps in proving our main result.

**Lemma 5.** Let  $G = (V_1 \cup V_2 \cup \dots \cup V_{n-m}, E)$  be a  $(n-m)$ -partite graph with  $1 < 2m \leq n$ ,  $n \geq 1$ ,  $|V_1| \geq |V_2| \geq \dots \geq |V_{n-m}|$  and  $|V_1| + |V_2| + \dots + |V_{n-m}| = n$ . Then

$$|E| \leq \frac{n^2 - n - 2m}{2}$$

In particular,

$$|E| = \frac{n^2 - n - 2m}{2}$$

if and only if  $G$  is a complete  $(n-m)$ -partite graph  $K_{|V_1|, |V_2|, \dots, |V_{n-m}|}$  with

$$|V_1| = |V_2| = \dots = |V_m| = 2, |V_{m+1}| = \dots = |V_{n-m}| = 1$$

*Proof.* We will prove this lemma by induction on  $t = n - m$ . If  $t = 1$  then it is easy to deduce that  $m = 1, n = 2$ . So the assertion holds. Now let  $t \geq 1$  and assume the assertion for smaller values of  $t$ . If  $|V_m| \geq 3$  then

$$|V_1| + |V_2| + \dots + |V_{n-m}| \geq 3m + (n - 2m) = n + m > n$$

a contradiction. Therefore,  $|V_m| \leq 2$ . Now we consider separately two cases.

*Case 1:* There exists  $i \in \{1, 2, \dots, m\}$  such that  $|V_i| = 2$ .

Consider the graph  $G' = G - V_i$ . It is easy to see that  $G'$  is a  $(t-1)$ -partite graph

$$G' = (V_1 \cup V_2 \cup \dots \cup V_{i-1} \cup V_{i+1} \cup \dots \cup V_{n-m}, E')$$

With  $|V_1| + |V_2| + \dots + |V_{i-1}| + |V_{i+1}| + \dots + |V_{n-m}| = n - 2$ . By the induction hypothesis,

$$|E'| \leq \frac{(n-2)^2 - (n-2) - 2(m-1)}{2}.$$

Then,

$$\begin{aligned} |E| &\leq |E'| + |V_i|(|V_1| + |V_2| + \dots + |V_{i-1}| + |V_{i+1}| + \dots + |V_{n-m}|) \\ &\leq \frac{(n-2)^2 - (n-2) - 2(m-1)}{2} + 2(n-2) \\ &= \frac{n^2 - n - 2m}{2} \end{aligned}$$

$$\text{If } |E| = \frac{n^2 - n - 2m}{2} \text{ then } |E'| = \frac{(n-2)^2 - (n-2) - 2(m-1)}{2}$$

and

$$|V_i|(|V_1| + |V_2| + \dots + |V_{i-1}| + |V_{i+1}| + \dots + |V_{n-m}|) = 2(n-2)$$

From the inductive hypothesis it is easy to deduce that  $G$  is a complete  $(n-m)$ -partite graph  $K_{|V_1|, |V_2|, \dots, |V_{n-m}|}$  with

$$|V_1| = |V_2| = \dots = |V_m| = 2, |V_{m+1}| = \dots = |V_{n-m}| = 1$$

*Case 2:*  $|V_i| \neq 2$  for every  $i \in \{1, 2, \dots, m\}$ .

If  $|V_1| = 1$  then  $|V_1| = |V_2| = \dots = |V_{n-m}| = 1$ . So,  $|V_1| + |V_2| + \dots + |V_{n-m}| = n - m$ . But by hypothesis we have  $|V_1| + |V_2| + \dots + |V_{n-m}| = n$ . It follows that  $m = 0$ , contradicts the assumption that  $1 < 2m \leq n$ . Thus,  $|V_1| \geq 3$ . From  $|V_m| \leq 2$  we can assume that there exists  $h \in \{1, 2, \dots, m\}$  such that  $|V_h| = 1$  and  $|V_{h-1}| \geq 3$ .

Let  $G_1 = K_{t_1, t_2, \dots, t_{n-m}}$  be a complete  $(n-m)$ -partite graph such that  $t_{h-1} = |V_{h-1}| - 1 \geq 2$ ,  $t_h = |V_h| + 1 = 2$  and  $t_i = |V_i|$  for every  $i \in \{1, 2, \dots, m\} \setminus \{h-1, h\}$ . Then,  $t_1 \geq t_2 \geq \dots \geq t_{n-m}$ ,  $t_1 + t_2 + \dots + t_{n-m} = n$  and there exists  $h \in \{1, 2, \dots, m\}$  such that  $t_h = 2$ . Similar argument to Case 1 we have,

$$|E(G_1)| \leq \frac{n^2 - n - 2m}{2}$$

On the other hand, we have

$$\begin{aligned} |E(G_1)| &= \sum_{1 \leq i < j \leq n-m} t_i t_j \\ &= \sum_{\substack{i, j \in \{1, 2, \dots, n-m\} \\ i, j \notin \{h-1, h\}}} t_i t_j + \sum_{\substack{i \in \{1, 2, \dots, n-m\} \\ i \notin \{h-1, h\}}} t_i t_{h-1} \\ &\quad + \sum_{\substack{i \in \{1, 2, \dots, n-m\} \\ i \notin \{h-1, h\}}} t_i t_h + t_{h-1} t_h \\ &= \sum_{\substack{i, j \in \{1, 2, \dots, n-m\} \\ i, j \notin \{h-1, h\}}} |V_i| |V_j| + \sum_{\substack{i \in \{1, 2, \dots, n-m\} \\ i \notin \{h-1, h\}}} |V_i| (|V_{h-1}| - 1) \\ &\quad + \sum_{\substack{i \in \{1, 2, \dots, n-m\} \\ i \notin \{h-1, h\}}} |V_i| (|V_h| + 1) + (|V_{h-1}| - 1)(|V_h| + 1) \\ &= \sum_{1 \leq i < j \leq n-m} |V_i| |V_j| + |V_{h-1}| - |V_h| - 1 \\ &\geq |E| + 1 \end{aligned}$$

From that we can deduce

$$|E| \leq |E(G_1)| - 1 < \frac{n^2 - n - 2m}{2}$$

Now, we will state and prove the main result of the paper.

**Theorem 6.** *The graph  $G^* = (V^*, E^*)$  is  $\chi$ -unique.*

*Proof.* Suppose that  $G' = (V', E')$  is a graph such that  $G' \sim G^*$ . Since Lemma 3 and (iii) of Lemma 4 we have

$$\chi(G') = \chi(G^*) = n - m$$

Suppose  $f$  is an  $(n-m)$ -coloring of graph  $G'$  using the colors  $1, 2, \dots, n-m$ . Put

$$V'_i = \{u \in V' \mid f(u) = i\}$$

for every  $i = 1, 2, \dots, n-m$ . By renaming the colors, we can assume that

$$|V'_1| \geq |V'_2| \geq \dots \geq |V'_{n-m}|$$

It is easy to see that  $G'$  is a  $(n-m)$ -partite graph  $(V'_1 \cup V'_2 \cup \dots \cup V'_{n-m}, E')$ . By Lemma 1, (i) and (ii) of Lemma 4 we have

$$V(G') = V(G^*) = n, E(G') = E(G^*) = \frac{n^2 - n - 2m}{2}$$

By Lemma 5, we have

$$|V'_1| = |V'_2| = \dots = |V'_m| = 2, |V'_{m+1}| = \dots = |V'_{n-m}| = 1$$

It follows that  $G' \cong G^*$ . Thus  $G^*$  is  $\chi$ -unique.

#### 4. CONCLUSION

The graph coloring problem is a difficult and interesting problem and is one of the central problems of graph theory. This problem also has many applications in the fields of Information Technology and Computer Science such as information security, channel distribution, etc.

The graph coloring problem has been studied for a long time and is still of interest to many mathematicians. Up to now, many profound and interesting results have been obtained. In this paper, we study the problem of coloring the complete graph class remove some disjoint edges. The main result obtained is to prove that this graph class is chromatically unique (Theorem 6). To obtain the above result, our main tool is to use the combinatorial method with a meticulous and scientific analysis of the structure of the graph class under

consideration. In the future, we will continue to study the problem of coloring other more general graph classes, hoping to obtain more profound results.

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