ATTRIBUTE REDUCTION BASED ON FUZZY WEIGHTED NEIGHBORHOOD ROUGH SETS

Pham Viet Anh[1,*], Tran Hung Cuong[2]

DOI: http://doi.org/ 10.57001/huih5804.2025.346

ABSTRACT

The neighborhood rough set model has been recognized as an effective tool and has been successfully applied to the problem of attribute reduction in numerical decision tables. However, this model and its various extensions still face significant limitations in reflecting the importance of individual condition attributes. Moreover, the traditional neighborhood rough set model assumes that all objects within a neighborhood granule contribute equally, despite their actual contributions to classification performance may vary. To address these shortcomings, we initially construct a new type of information granule based on the integration of condition attribute weights and object weights. Based on these granules, we propose a fuzzy weighted neighborhood rough set (FWNRS) model and develop a new uncertainty classification measure to define an effective reduct. Finally, we design an attribute reduction algorithm for decision tables that can be applied across various data scenarios.

Keywords: *Decision table, Reduct, Neighborhood rough sets.*

[1]HaUI Institute of Technology, Hanoi University of Industry, Vietnam

[2]School of Information and Communications Technology, Hanoi University of Industry, Vietnam

[*]Email: anhpv@haui.edu.vn

Received: 07/7/2025

Revised: 15/9/2025

Accepted: 28/9/2025

1. INTRODUCTION

In recent years, attribute reduction on decision tables has emerged as a critical issue, attracting significant attention from the research community. The main goal of this process is to retain a subset of key conditional attributes in order to enhance the effectiveness of classification models. It is well known that the rough set model proposed by Pawlak [1] has been effectively applied to the problem of attribute reduction on decision tables with discrete and categorical attributes. However, this model has significant limitations when dealing with decision tables containing numerical attributes due to its use of a strict relation. Therefore, extending this model has become a highly promising research area.

The neighborhood rough set model has recently gained wide recognition and is considered an effective solution for addressing the problem of attribute reduction in decision tables containing numerical/continuous attributes. Instead of employing indiscernibility relations as in the rough set model, the neighborhood rough set model utilizes neighborhood relations. Consequently, each object in the universe is represented by a neighborhood class consisting of objects that are neighbors of the given object within a radius of δ [2]. Based on the neighborhood rough set model, many effective attribute reduction methods were proposed. Particularly, Hu et al. constructed some measures including the neighborhood decision error rate [3]. Based on the neighborhood discrimination index, Wang et al. [4] proposed an algorithm for finding an optimal reduct through the conditional discrimination index. Subsequently, Sun et al. [5] presented a new heuristic algorithm for attribute reduction, which incorporates the neighborhood tolerance dependency joint entropy to efficiently handle mixed and incomplete datasets. Expanding upon other extensions of neighborhood rough sets, Yang et al. [6] developed pseudo-label neighborhood rough sets and proposed a heuristic algorithm that utilizes the pseudo-label conditional entropy measure to handle uncertain data. Zhang et al. [7] developed a conditional neighborhood conbination information entropy measure to handle diverse data in feature extraction. On the other hand, Yang et al. [8] approached the problem through distance metric learning, optimizing the structure of information granules and proposing two feature selection algorithms. Additionally, Wang et al. [9] introduced *k*-nearest neighborhood rough sets to address this type of heterogeneous data. Although algorithms based on the neighborhood rough set space have been developed in different ways [10-12] they still face certain limitations due to the following issues:

1) The first issue stems from the fact that the neighborhood rough set model does not accurately reflect the impact of each conditional attribute on the decision attributes.

2) The second issue is that some measures developed based on this model focus only on the objects within an information granule while ignoring other objects, even though these objects also contribute to the classification capability.

3) The third limitation is that the neighborhood rough set model assumes the roles of objects within an information granule to be the same, although in reality these objects have different distributions.

To address the aforementioned issues, this study initially proposes a new type of information granule constructed using attribute weights and object weights. Based on these information granules, we introduce a new concept called the fuzzy weighted neighborhood rough set model. This model is capable of providing a more detailed assessment of the role of each object within an information granule compared to traditional neighborhood rough set models. Accordingly, we construct an uncertainty classification measure and define a new efficient reduct. Finally, we design an attribute reduction algorithm based on a filter approach to extract an optimal subset of attributes.

The main content of this paper is organized as follows. Section 2 presents the relevant knowledge concerning decision tables and the neighborhood rough set theory. Section 3 describes the fuzzy weighted neighborhood rough set model and the uncertainty classification measure. Section 4 introduces the attribute reduction algorithm based on the fuzzy weighted neighborhood rough set model, and Section 5 provides some discussions as well as future research directions.

2. BASIC NOTIONS

This section provides a summary of concepts related to the neighborhood rough set model. This theory is a popular tool often used for the attribute reduction problem.

In reality, data is often organized as a *decision table* $DS=(U,C\cup D)$, where U is referred to as the universe set and contains objects, C and D are the sets of *condition attributes* and *decision attributes*, respectively, satisfying $C\cap D=\emptyset$. Then, with $u\in U$ and $a\in C\cup D$, a(u) is considered the attribute value of u on attribute a.

Example 1. Let $S=(U,C\cup D)$, where $U=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ as Table 1.

Table 1. A decision table

U	a_1	a_2	a_3	a_4	a_5	D
u_1	0.72	0.93	0.65	0.52	0.69	Yes
u_2	0.43	0.92	0.52	0.44	0.57	No
u_3	0.21	0.64	0.85	0.62	0.21	Yes
u_4	0.95	0.82	0.21	0.81	0.94	No
u_5	0.45	0.72	0.28	0.87	0.18	Yes

Let $DS=(U,C\cup D)$ be a decision table, $A\subseteq C$ be a subset of attributes, and $u,v\in U$. The distance between the two objects u and v with respect to the attribute set A, denoted as $\Delta_A(u,v)$, is defined as follows:

$$\Delta_A(u,v)=\sqrt[p]{\sum_{a\in A}|a(u)-a(v)|^p} \quad (1)$$

where $\Delta_A(u,v)$ is called the Manhattan distance if p = 1, the Euclidean distance if p = 2, and the Chebyshev distance if p = ∞.

Suppose that δ is a neighborhood radius with a value in the range [0, 1]. Then, a binary relation R_A^δ, called a *neighborhood relation* on U from the attribute set A, is defined as follows:

$$R_A^\delta=\{(u,v)\in U\times U:\Delta_A(u,v)\le\delta\} \quad (2)$$

Clearly, the neighborhood relation represents the similarity or dissimilarity between objects in the universe. Based on this relation, $[u]_A^\delta=\{v\in U:(u,v)\in R_A^\delta\}$ is defined as the *neighborhood information granule* of object *u* induced by the attribute subset A. It is easy to see that the neighborhood information granule $[u]_A^\delta$ is an ordinary set that satisfies $[u]_A^\delta\subseteq U$. Then, by considering all objects in the universe space, we can obtain a family of neighborhood information granules, denoted as $U/R_A^\delta=\{[u]_A^\delta:u\in U\}$, which is referred to as a *neighborhood cover* induced by the attribute set A.

Based on the concept of a neighborhood information granule, for a set of objects $X \subseteq U$, the definitions of the *upper approximation* $\overline{N_A}(X)$ and the *lower approximation* $\underline{N_A}(X)$ of U are presented as follows.

$$\overline{N_A}(X) = \left\{u \in U : [u]_A^\delta \cap X \neq \emptyset\right\} \quad (3)$$

and

$$\underline{N_A}(X) = \left\{u \in U : [u]_A^\delta \subseteq X\right\} \quad (4)$$

3. FUZZY WEIGHTED NEIGHBORHOOD ROUGH SETS

3.1. Fuzzy weighted neighborhood information granules

This section starts with the following definition of the *generalized weighted distance*. Given an attribute A and two objects u, v, the generalized weighted distance between u and v with respect to A, denoted $\Delta_A^\omega(u,v)$, is determined by

$$\Delta_A^\omega(u,v) = \sqrt[p]{\sum_{a \in A} \omega(a).|a(u) - a(v)|^p} \quad (5)$$

where ω is the weight of attribute $a \in C$. Then, a generalized attribute weighted neighborhood information granule of object u is defined as:

$$[u]_A^{\delta,\omega} = \{v \in U : \Delta_A^\omega(u,v) \leq \delta\} \quad (6)$$

Normally, we consider the objects in $[u]_A^{\delta,\omega}$ with the same role degree when making decisions for object u. However, each of these objects may play a different role in evaluating u. Therefore, considering each object's degree of importance, also known as weight, is vital. Next, we shall more closely consider this information granule by assigning weights to each object within it. Suppose that $[u]_A^{\delta,\omega} = \left\{u_1, u_2, \ldots, u_{\left|[u]_A^{\delta,\omega}\right|}\right\}$, we can assign $\omega_A^u(u_1)$, $\omega_A^u(u_2), \ldots, \omega_A^u\left(u_{\left|[u]_A^{\delta,\omega}\right|}\right)$ as the weights of objects in $[u]_A^{\delta,\omega}$, in which $\omega(u_i) \in [0,1]$ for all $1 \leq i \leq \left|[u]_A^{\delta,\omega}\right|$. The greater the weight value of $\omega_A^u(u_i)$, the more important the object u_i is in evaluating object u. Combining $[u]_A^{\delta,\omega}$ with its weights, we propose a novel information granule known as a fuzzy weighted neighborhood information granule, which has the following definition

$$[\tilde{u}]_A^{\delta,\omega} = \left\{\omega_A^u(u_1), \omega_A^u(u_2), \ldots, \omega_A^u\left(u_{|U|}\right)\right\} \quad (7)$$

Clearly, $[\tilde{u}]_A^{\delta,\omega}$ can be considered as a fuzzy set on U if the objects in $[\tilde{u}]_A^{\delta,\omega}$ have the membership function values as their weights, and other objects have the membership function values by 0. Therefore, in the subsequent sections of the paper, operations related to $[\tilde{u}]_A^{\delta,\omega}$ can also be performed in the same way as for a fuzzy set. Then, we denote $\tilde{G}(A) = \left\{[\tilde{u}]_A^{\delta,\omega} : u \in U\right\}$ the family of all fuzzy weighted neighborhood information granules. In this paper, the weight of each attribute $a \in C$ is determined by $\omega(a) = \frac{1}{1+\sigma(a)}$, where σ(a) is the standard deviation of the values of objects for condition attribute a. The weight of each object $v \in [\bar{u}]_A^{\delta,\omega}$ is calculated as follows:

$$\omega_A^u(v) = \frac{1}{1 + \Delta_A^\omega(u,v)} \cdot \frac{\left|[u]_C^{\delta,\omega} \cap [u]_D\right|}{\left|[u]_C^{\delta,\omega}\right|} \quad (8)$$

where $[u]_D = \bigcap_{d \in D}\{v \in U : d(u) = d(v)\}$ is an equivalence class of object u on D.

Formula (8) is the product of two components. The first component is regarded as the relative distance between two objects u and v. As this distance increases, the weight of object v in $[u]_A^{\delta,\omega}$ decreases, i.e. the role of v in the granule is diminished. The second component measures the amount of information that u contributes to the decision class D. If this amount of information is small, the roles of all objects in $[u]_A^{\delta,\omega}$ are also weakened.

Example 2. With δ = 0.6, we determine the fuzzy weighted neighborhood information family $\tilde{G}(C)$ as follows.

- The partition of the attribute decsion: $U/D = \{\{u_1, u_3, u_5\}, \{u_2, u_4\}\}$

-Determine the weight of each condition attribute a by Equation $\omega(a) = \frac{1}{1+\sigma(a)}$:

$\omega(a_1) = 1.26$, $\omega(a_2) = 1.11$, $\omega(a_3) = 1.24$, $\omega(a_4) = 1.16$, $\omega(a_5) = 1.29$

- Compute the weighted distance between the objects on C:

$$\Delta_C^\omega(u_1,u_2) = \max\left\{\begin{matrix} 1.26 \times |0.72 - 0.43|, \\ 1.11 \times |0.93 - 0.93|, \ldots, \\ 1.29 \times |0.69 - 0.57| \end{matrix}\right\} = 0.54$$

Similarly, we obtain:

$\Delta_C^\omega(u_1,u_3)=0.86$, $\Delta_C^\omega(u_1,u_4)=0.54$,

$\Delta_C^\omega(u_1,u_5)=0.35$, $\Delta_C^\omega(u_2,u_3)=0.71$,

$\Delta_C^\omega(u_2,u_4)=0.53$, $\Delta_C^\omega(u_2,u_5)=0.70$,

$\Delta_C^\omega(u_3,u_4)=0.76$, $\Delta_C^\omega(u_3,u_5)=0.85$,

$\Delta_C^\omega(u_4,u_5)=0.89$.

- Determine the weighted neighborhood information granules on C:

$[u_1]_C^{0.6,\omega}=\{u_1,u_2,u_4,u_5\}$, $[u_2]_C^{0.6,\omega}=\{u_2,u_4\}$,

$[u_3]_C^{0.6,\omega}=\{u_3\}$, $[u_4]_C^{0.6,\omega}=\{u_1,u_2,u_4\}$,

$[u_5]_C^{0.6,\omega}=\{u_1,u_5\}$.

- Compute fuzzy weighted neighborhood granules with respect to C:

$$\omega_C^{u_1}(u_1)=\frac{1}{1+\Delta_C^\omega(u_1,u_1)}.\frac{\left|[u_1]_C^{\delta,\omega}\cap[u_1]_D\right|}{\left|[u_1]_C^{\delta,\omega}\right|}$$

$$=\frac{1}{1+0}\times\frac{2}{4}=0.5$$

Similarly, we obtain:

$$\omega_C^{u_1}(u_2)=\frac{1}{1+0.54}\times\frac{2}{4}=0.32,$$

$$\omega_C^{u_1}(u_3)=\frac{1}{1+0.86}\times\frac{2}{4}=0.27,$$

$$\omega_C^{u_1}(u_4)=\frac{1}{1+0.54}\times\frac{2}{4}=0.32,$$

$$\omega_C^{u_1}(u_5)=\frac{1}{1+0.35}\times\frac{2}{4}=0.37.$$

Hence, we have:

$$[\tilde{u}_1]_C^{0.6,\omega}=\left\{\frac{0.5}{u_1},\frac{0.32}{u_2},\frac{0.27}{u_3},\frac{0.32}{u_4},\frac{0.37}{u_5}\right\}.$$

- Similarly, we also obtain $[\tilde{u}_2]_C^{0.6,\omega}$, $[\tilde{u}_3]_C^{0.6,\omega}$, $[\tilde{u}_4]_C^{0.6,\omega}$, $[\tilde{u}_5]_C^{0.6,\omega}$ and fuzzy weighted neighborhood information family $\tilde{G}(C)$.

$$\tilde{G}(C)=\begin{bmatrix}0.50 & 0.32 & 0.27 & 0.32 & 0.37\\ 0.65 & 1.00 & 0.91 & 0.65 & 0.59\\ 0.54 & 0.58 & 1.00 & 0.57 & 0.54\\ 0.43 & 0.44 & 0.38 & 0.67 & 0.35\\ 0.74 & 0.59 & 0.54 & 0.53 & 1.00\end{bmatrix}$$

3.2. Basic concepts of FWNRSs

Under the fuzzy weighted information granules framework, we propose a new rough set model named *fuzzy weighted neighborhood rough sets* (FWNRSs).

Given a decision table $DS=(U,C\cup D)$, an attribute subset $A\subseteq C$ and an object subset $X\subseteq U$. The *upper and lower approximations* of X based on the fuzzy weighted neighborhood information granules with respect to A are respectively determined as follows:

$$\underline{FW_A}(X)=\left\{u\in U:\frac{\left|[\tilde{u}]_A^{\delta,\omega}\cap X\right|}{\left|[\tilde{u}]_A^{\delta,\omega}\right|}\geq\alpha\right\} \quad (9)$$

and

$$\overline{FW_A}(X)=\left\{u\in U:\frac{\left|[\tilde{u}]_A^{\delta,\omega}\cap X\right|}{\left|[\tilde{u}]_A^{\delta,\omega}\right|}\geq\beta\right\} \quad (10)$$

where $0\leq\beta\leq\alpha\leq1$.

Then, the boundary of X with respect to $\tilde{G}(A)$ is defined as

$$FBN_A(X)=\overline{FW_A}(X)-\underline{FW_A}(X) \quad (11)$$

Next, we can construct the *fuzzy weighted positive region* and *fuzzy weighted boundary* of D with respect to A as follows:

$$FPOS_A(D)=\bigcup_{X\in U/D}\underline{FW_A}(X) \quad (12)$$

and

$$FBN_A(D)=\overline{FW_A}(D)-\underline{FW_A}(D) \quad (13)$$

where $\overline{FW_A}(D)=\bigcup_{X\in U/D}\overline{FW_A}(X)$ and $\underline{FW_A}(D)=\bigcup_{X\in U/D}\underline{FW_A}(X)$.

Let $S=(U,C\cup D)$ be a decision table, and A be an attribute subset of C. The *fuzzy weighted dependency degree* of D to A is determined by

$$\tilde{\gamma}(A,D)=\frac{|FPOS_A(D)|}{|U|} \quad (14)$$

Proposition 1. Given a decision table $S=(U,C\cup D)$ of FWNRSs, the following properties hold:

1) If $\beta=0$ then $\overline{FW_A}(D)=U$;

2) If $\beta=0$ then $FBN_A(D)=U-FPOS_A(D)$;

3) $FBN_A(D) \cap FPOS_A(D) = \emptyset$;

4) $FBN_A(D) \cup FPOS_A(D) = \overline{FW_A}(D)$.

Proof. 1) Based on Equation 10 and β = 0, if for any $v \in X$, then $v \in \overline{FW_A}(X)$. Hence, $X \subseteq \overline{FW_A}(X)$ for any $X \in U/D$. Thus, $\bigcup_{X \in U/D} X \subseteq \bigcup_{X \in U/D} \overline{FW_A}(X) = \overline{FW_A}(D)$. Meanwhile, $\bigcup_{X \in U/D} X = U$, hence $U \subseteq \overline{FW_A}(D)$. Additionally, $\overline{FW_A}(D) \subseteq U$. Therefore, $\overline{FW_A}(D) = U$.

2) Based on Equation 13, we have $FBN_A(D) = U - \underline{FW_A}(D)$. Additionally, $\underline{FW_A}(D) = \bigcup_{X \in U/D} \underline{FW_A}(X) = FPOS_A(D)$. Therefore, $FBN_A(D) = U - FPOS_A(D)$.

3) From Equation 13, $FBN_A(D) = \overline{FW_A}(D) - \underline{FW_A}(D) = \overline{FW_A}(D) - FPOS_A(D)$. Therefore, $FBN_A(D) \cap FPOS_A(D) = \emptyset$.

4) We have

$$FBN_A(D) = \overline{FW_A}(D) - \underline{FW_A}(D) = \overline{FW_A}(D) - FPOS_A(D).$$

Therefore, $FBN_A(D) \cup FPOS_A(D) = \overline{FW_A}(D)$.

Proposition 2. Let $S = (U, C \cup D)$ be a decision table and $A, B \subseteq C$. If $A \subseteq B$, then

1) $\forall u \in U$, $[\tilde{u}]_B^{\delta,\omega} \subseteq [\tilde{u}]_A^{\delta,\omega}$;

2) If β = 0, then $\forall X \subseteq U$, $\overline{FW_B}(X) \subseteq \overline{FW_A}(X)$;

3) If α = 1, then $\forall X \subseteq U$, $\underline{FW_A}(X) \subseteq \underline{FW_B}(X)$;

4) If α = 1, then $FPOS_A(D) \subseteq FPOS_B(D)$ and $\tilde{\gamma}(A,D) \leq \tilde{\gamma}(B,D)$.

Proof.

1) According to Equation 5 and $A \subseteq B$, we have $\sqrt{\sum_{a \in A} \omega(a).(a(u) - a(v))^p} \leq \sqrt{\sum_{a \in B} \omega(a).(a(u) - a(v))^p}$, for all $u, v \in U$. Therefore, for all $u \in U$, $[u]_B^{\delta,\omega} \subseteq [u]_A^{\delta,\omega}$, it follow that $[\tilde{u}]_B^{\delta,\omega} \subseteq [\tilde{u}]_A^{\delta,\omega}$.

2) From β = 0, for any $u \in \overline{FW_B}(X)$, we have $\frac{\left|[\tilde{u}]_B^{\delta,\omega} \cap X\right|}{\left|[\tilde{u}]_B^{\delta,\omega}\right|} \geq 0$, which implies that $\left|[\tilde{u}]_B^{\delta,\omega} \cap X\right| \geq 0$. Meanwhile, according to Proposition 2.1, since $A \subseteq B$, we obtain $[\tilde{u}]_B^{\delta,\omega} \subseteq [\tilde{u}]_A^{\delta,\omega}$. Thus, $\left|[\tilde{u}]_A^{\delta,\omega} \cap X\right| \geq 0$ and $u \in \overline{IW_A}(X)$. Hence, $\overline{FW_B}(X) \subseteq \overline{FW_A}(X)$.

3) From α = 1, for any $u \in \underline{FW_A}(X)$, we have $\frac{\left|[\tilde{u}]_A^{\delta,\omega} \cap X\right|}{\left|[\tilde{u}]_A^{\delta,\omega}\right|} \geq 1$, which implies that $[\tilde{u}]_A^{\delta,\omega} \subseteq X$. Meanwhile, according to Proposition 2.1, since $A \subseteq B$, we obtain $[\tilde{u}]_B^{\delta,\omega} \subseteq [\tilde{u}]_A^{\delta,\omega}$. Thus, $[\tilde{u}]_B^{\delta,\omega} \subseteq X$ và $\frac{\left|[\tilde{u}]_B^{\delta,\omega} \cap X\right|}{\left|[\tilde{u}]_B^{\delta,\omega}\right|} = 1$. Therefore, $u \in \underline{FW_B}(X)$ and $\underline{FW_A}(X) \subseteq \underline{FW_B}(X)$.

4) Based on Proposition 2.3, we always obtain :

$$FPOS_A(D) = \bigcup_{x \in U/D} \underline{FW_A}(X) \subseteq \bigcup_{x \in U/D} \underline{FW_B}(X) = FPOS_B(D).$$

Hence, $\tilde{\gamma}(A,D) \leq \tilde{\gamma}(B,D)$.

4. ATTRIBUTE REDUCTION WITH FWNRSs

This section introduces a key application of FWNRSs in attribute reduction for decision tables. The objective is to identify a minimal subset of attributes, referred to as a reduct, that preserves the essential information of the decision table, equivalent to that provided by the full set of attributes. Specifically, we define a reduct from the perspective of the dependency degree in the decision table.

Definition 1. Given a decision table $S = (U, C \cup D)$, an attribute subset A is called a $\tilde{\gamma}$-reduct of C based on the fuzzy weighted dependency degree if A satisfies the following conditions:

1) $\tilde{\gamma}(A,D) = \tilde{\gamma}(C,D)$,

2) $\forall a \in A$, $\tilde{\gamma}(A \setminus \{a\}, D) < \tilde{\gamma}(A,D)$.

It is evident that a $\tilde{\gamma}$-reduct in Definition 1 is the minimal subset of attributes that preserves the consistency factor of all attributes in the decision table. In other words, this reduct contains only the objects in the positive region of the decision table and ignores the others. However, in cases where the decision table is inconsistent, the number of objects outside the positive region will be larger. These objects are considered unclassifiable with certainty, and ignoring them in the computation process can significantly affect the quality of the obtained reduct. Therefore, we define another type of reduct to handle all objects in the universe. First, we construct a measure, referred to as the uncertainty classification degree of objects in the universe, based on a given attribute subset.

Definition 2. Given a decision table $S=(U,C\cup D)$ and an attribute subset $A\subseteq C$, the uncertainty classification degree of objects in U employing the attribute subset A, denoted $\tilde{\eta}(A,D)$, is defined as

$$\tilde{\eta}(A,D)=\frac{1}{|U|}\sum_{u\in U}\frac{\sum_{v\in U}\max\{0,\omega_A^u(v)-\omega_D^u(v)\}}{\left|[\tilde{u}]_A^{\delta,\omega}\right|} \quad (15)$$

It is evident that $0\le\tilde{\eta}(A,D)\le(|U|-1)/|U|$. Intuitively, the classification ability of the attribute subset A increases as the uncertainty classification degree $\tilde{\eta}(A,D)$ decreases, and vice versa. In the case where $\tilde{\eta}(A,D)=0$, every object in U can be classified with certainty based on the attribute subset A. Based on this measure, we further redefine a new reduct as the basis for developing an attribute reduction algorithm for the decision table.

Example 3. From fuzzy weighted neighborhood information family $\tilde{G}(C)$, we compute $\tilde{\eta}(C,D)$ as follows.

$$\tilde{\eta}(C,D)=\frac{1}{5}\times\frac{(0.32+0.32)+(0.65+0.91+0.59)+(0.58+0.57)}{5}$$
$$+\frac{1}{5}\times\frac{(0.43+0.38+0.35)+(0.59+0.53)}{5}=0.27$$

Definition 3. Given a decision table $S=(U,C\cup D)$, an attribute subset $A\subseteq C$ is called a $\tilde{\eta}$-reduct of C relative to D if A satisfies the following

1) $\tilde{\eta}(A,D)=\tilde{\eta}(C,D)$,

2) $\forall a\in A$, $\tilde{\eta}(A\setminus\{a\},D)\neq\tilde{\eta}(A,D)$.

Clearly, a $\tilde{\eta}$-reduct is a subset that preserves the certainty degree of all objects in the universe. Therefore, the $\tilde{\eta}$-reduct has better generalization ability than the $\tilde{\gamma}$-reduct. Based on this definition, we present the attribute importance for selecting the key attributes of the decision table.

Definition 4. Given a decision table $S=(U,C\cup D)$, an attribute subset $A\subseteq C$ and an attribute $a\in C\setminus A$, the significance measure of the attribute a with respect to A, denoted Sig(a, A), is detemined by

$$Sig(a,A)=\tilde{\eta}(A,D)-\tilde{\eta}(A\cup\{a\},D) \quad (16)$$

The significance of any attribute for an attribute subset is exactly the change of the certainty degree when that attribute is added to the attribute subset. Intuitively, we can see that the change of the certainty degree, i.e. the value of Sig(a, A), is bigger, the attribute *a* will be more vital. This is because the uncertainty classification does not satisfy the monotonicity property with respect to the size of the attribute subset. In this case, the attribute is considered noisy and deemed unimportant in the decision table. In the case of attributes with positive significance values, these attributes are considered meaningful and make a significant contribution to the decision table. From this definition, we will design an algorithm to extract a subset of attributes from the decision table. The algorithm begins with an empty set of attributes and then iteratively adds the attributes with the highest significance in each iteration until the stopping condition is satisfied. The algorithm is specifically presented in the pseudocode in Algorithm 1.

Algorithm 1. Attribute Reduction with FWNRSs (ARFWNR)		
Input: a decision table $S=(U,C\cup D)$ and a neighborhood radius δ **Output:** a reduct red		
1	compute the weight of each attribute $a\in C$	
2	compute $\tilde{\eta}(C,D)$ by Equation 15	
3	**for** $a\in C$ **do**	
4		compute $\tilde{G}(\{a\})$ by Equation 8
5		compute $\tilde{\eta}(\{a\},D)$ by Equation 15
6	**end for**	
7	$red=\{a_0\}$ which satisfies: $\tilde{\eta}(\{a_0\},D)=\min_{a\in C}\tilde{\eta}(\{a\},D)$	
8	**while** $\tilde{\eta}(red,D)\neq\tilde{\eta}(C,D)$ **do**	
9		compute $Sig(a,red)$, for all $a\in C\setminus red$ by Equation 16
10		select a_0 which satisfies: $Sig(a_0,red)=\max_{a\in C\setminus red}\{Sig(a,red)\}$
11		$red\leftarrow red\cup\{a_0\}$
12	**end while**	
13	**return** red	

To evaluate the computational complexity of Algorithm 1, suppose that $|C|,|U|$ represent the number of condition attributes and the number of objects in the decision table, respectively. It is easy to see that the complexity of the algorithm when calculating the weights of each condition attribute is $O(|C|.|U|)$. The execution time of the algorithm in Step 2 is $O\left(|C|.|U|^2\right)$, which is also the computational complexity in the *for* loop from Step 3 to Step 6. In the *while* loop, whenever an added attribute does not satisfy the property of a reduct, the remaining attributes will continue to be considered in order to select the one with the greatest importance. In that case, the complexity from Step 8 to Step 12 is $O\left(|C|^2.|U|^2\right)$. Therefore, the overall complexity of the entire algorithm is $O\left(|C|^2.|U|^2\right)$.

5. CONCLUSION

In this study, we initially proposed a new type of neighborhood information granule, which is composed of the weights of condition attributes and the weights of objects within the granule. These information granules were then used to construct Fuzzy Weighted Neighborhood Rough Sets (FWNRSs). Based on this, we presented several important properties of FWNRSs and proposed a new measure to evaluate all objects in the universe. From this measure, we defined an effective reduct and designed an attribute reduction algorithm. In the future, we will continue to develop new models as a foundation for designing algorithms capable of handling various data scenarios.

REFERENCES

[1]. Pawlak Z., "Rough sets," *International Journal of Computer & Information Sciences,* 11, 643–666, 1982.

[2]. Hu Q., Liu J., Yu D., "Mixed feature selection based on granulation and approximation," *Knowledge-Based System*, 21, 294–304, 2008.

[3]. Hu Q., Pedrycz W., Yu D., Lang J., "Selecting discrete and continuous features based on neighborhood decision error minimization," *IEEE Transactions on Systems, Man, and Cybernetics, Part B Cybernetics*, 40, 137–150, 2010.

[4]. Wang C., Hu Q., Wang X., Chen D., Qian Y., Dong Z., "Feature selection based on neighborhood discrimination index," *IEEE Transactions on Neural Networks and Learning Systems,* 29, 2986–2999, 2018.

[5]. Sun L., Wang L., Qian Y., Xu J., Zhang S., "Feature selection using lebesgue and entropy measures for incomplete neighborhood decision systems," *Knowledge-Based System,* 186, 104942, 2019.

[6]. Yang X., Liang S., Yu H., Gao S., Qian Y., "Pseudo-label neighborhood rough set: Measures and attribute reductions," *International Journal of Approximate Reasoning*, 105, 112-129, 2019.

[7]. Zhang P., Li T., Yuan Z., Luo C., Liu K., Yang X., "Heterogeneous Feature Selection Based on Neighborhood Combination Entropy," *IEEE Transactions on Neural Networks and Learning Systems,* 35, 1–14, 2022.

[8]. Yang X., Chen H., Li T., Wan J., Sang B., "Neighborhood rough sets with distance metric learning for feature selection," *Knowledge-Based System*, 224, 107076, 2021.

[9]. Wang C., Shi Y., Fan X., Shao M., "Attribute reduction based on k-nearest neighborhood rough sets," *International Journal of Approximate Reasoning*, 106, 18-31, 2019.

[10]. Hu M., Tsang ECC., Gou Y., Chen D., Xu W., "Attribute reduction based on overlap degree and k-nearest-neighbor rough sets in decision information systems," *Information Sciences,* 584, 301-324, 2022.

[11]. Chen H., Li M., Fan X., Lou X., "Feature selection for imbalanced data based on neighborhood rough sets," *Information Sciences*, 483, 1–20, 2019.

[12]. Anh P. V., Giang N. L., Thuy N. N., Dung L. V., Quan P. H., "Hybrid filter-wrapper attribute reduction method with the uncertainty classification degree," *Advances in Data Science and Optimization of Complex Systems - Proceedings of the International Conference on Applied Mathematics and Computer Science,* 1569, 298–309, 2025.