FREE VIBRATION ANALYSIS OF THE FUNCTIONALLY GRADED STEPPED CYLINDRICAL SHELLS USING THE CONTINUOUS ELEMENT METHOD

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ABSTRACT

This research presents a continuous element model for solving vibration problems of functional graded (FG) stepped cylindrical shells (SCS). Based on the First Order Shear Deformation Theory (FSDT) and the equations of the FG cylindrical shells, the dynamic stiffness matrix is obtained for each segment of the shell having constant thickness. The interesting assembly procedure of continuous element method (CEM) is employed for joining those segments in order to analyze the dynamic behavior of the FG cylindrical. Free vibrations of different configurations of FG stepped cylindrical shells are examined. Effects of the power-law exponent *p*, FG materials properties on the free vibration of FG stepped cylindrical shells are also presented. The advantages of Continuous Element model are confirmed in terms of precision as well as its performance when dealing with complex FG structures.

Keywords: *Stepped cylindrical shells, FG cylindrical shells, Continuous Element Method, Dynamic stiffness matrix, Vibration of FG shells.*

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1. INTRODUCTION

Cylindrical shells made of functional graded materials (FGMs) are widely used in modern engineering structures such as tunnels, storage tanks, pressure vessels, water ducts, pipelines and casing pipes and in other applications. Therefore, dynamic analysis of shells is important for the safety and stability of those structures. The dynamic analyses of FGM cylindrical shells have been studied in recent years and many significant results are obtained. Based on the Flügge thin shell theory, Zhang et al [1] presented exact solutions for the vibration of circular cylindrical shells with step-wise thickness variations in the axial direction. Tornabene [2] focuses on the dynamic behavior of moderately thick functionally graded conical, cylindrical shells and annular plates by using the Generalized Differential Quadrature method. Qu et al. [3] developed an efficient domain decomposition algorithm for free and forced vibration analysis of the uniform and stepped conical shells. Su et al. [4] applied the Rayleigh-Ritz method and FSDT to study the free vibrations of FM graded cylindrical, conical shells and annular plates with general boundary conditions.

In recent years, Casimir et al [5] have succeeded in building the DSM for thick isotropic plate and shells of revolution. Recently, Thinh et al proposed the Continuous Element Method (CEM) or Dynamics Stiffness Method (DSM) [6] has been proposed based on the FSDT is proposed for free vibration analysis of thick cross-ply laminated composite cylindrical shells. Nam et al. [7] presented a continuous element model for solving vibration problems of stepped composite cylindrical shells surrounded by Pasternak foundations with various boundary conditions. Vinh et al. [8-9] present a new Continuous Element for analyzing dynamic behavior of stepped FG conical shells and annular plates. In this work, a powerful assembly procedure has been presented for constructing new dynamic stiffness matrix of stepped FG for structures. The continuous element formulations here are established based on the analytical solution of differential equations for structures giving high precision results

The main purpose of this paper is to present a new Continuous Element model to analyze the dynamic behavior of the FG stepped cylindrical shells with various material characteristics. Based on the assembly procedure of single continuous elements, the dynamic stiffness matrix of complex stepped cylindrical shells is established. In this research, the influences of different parameters are studied in detail such as: the power-law exponent p, FGM properties. The Continuous Element method can easily be used to analyze complex structures and it assured giving the high precise solutions.

2. THEORETICAL FORMULATIONS

2.1. Description of the model

Let's investigate the FGM cylindrical shell with (x, θ, z) coordinates, as shown in Fig. 1. Where x is the coordinate long the cylindricals' generators with the origin placed at the middle of the generators, θ is the circumferential coordinate, and z is the perpendicular to the cylindricals' surfaces. R is the mid-surface radius of the cylinder; L are lengths of the cylinder respectively, thicknesses h.

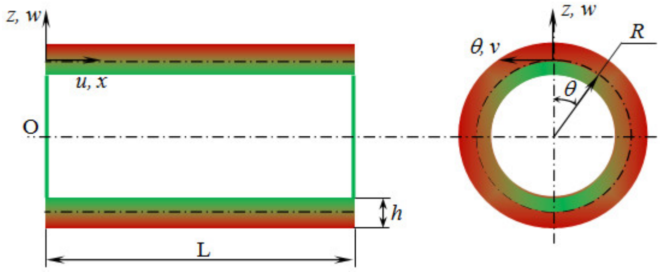

Fig. 1. Geometry of FGM cylindrical shell

Typically, FGM shells made from a mixture of two material phases. In this paper, it is assumed that the FGM shells are made of a mixture of ceramic and metal. Young's modulus E(z), density ρ(z) and Poisson's ratio μ(z) are assumed to vary continuously through the shells thickness and can be expressed as a linear combination:

$$E(z)=(E_c-E_m)V_c+E_m,\mu(z)=(\mu_c-\mu_m)V_c+\mu_m,\rho(z)=(\rho_c-\rho_m)V_c+\rho_m \quad (1)$$

in which the subscripts c and m represent the ceramic and metallic constituents, respectively, and the volume fraction V_c follows two general four-parameter power-law distributions [2, 4]:

$$\begin{aligned}
&FGM_{I(a/b/c/p)}: \quad V_c=\left[1-a\left(\frac{1}{2}+\frac{z}{h}\right)+b\left(\frac{1}{2}+\frac{z}{h}\right)^c\right]^p\\
&FGM_{II(a/b/c/p)}: \quad V_c=\left[1-a\left(\frac{1}{2}-\frac{z}{h}\right)+b\left(\frac{1}{2}-\frac{z}{h}\right)^c\right]^p
\end{aligned} \quad (2)$$

where p is a positive real number ($0\le p\le\infty$) and a, b, c dictate the material variation profile through the FG shell thickness. It is assumed that $V_c+V_m=1$. When $p=0$ or $p=\infty$ the FGM material becomes the homogeneous isotropic material, as:

$$\begin{aligned}
&p=0\to V_c=1,\ \ V_m=0\to E(z)=E_c,\ \ \mu(z)=\mu_c,\ \ \rho(z)=\rho_c\\
&p=\infty\to V_c=0,\ \ V_m=1\to E(z)=E_m,\ \ \mu(z)=\mu_m,\ \ \rho(z)=\rho_m
\end{aligned} \quad (3)$$

The volume fraction V_c varies depending on the coefficients a, b, c and the volume exponent p. When the volume exponent p = 1, the volume ratio of ceramic and metal changes linearly. when p is different from 1, then the volume fraction Vc varies according to different laws are available in [2].

2.2. Kinematic relations and stress resultants

The displacement components of an arbitrary point in the FG shell for the first-order shear deformation theory are expressed as given below [6]:

$$\begin{aligned}
&u(x,\theta,z,t)=u_0(x,\theta,t)+z\varphi_x(x,\theta,t),\\
&v(x,\theta,z,t)=v_0(x,\theta,t)+z\varphi_s(x,\theta,t),\\
&w(x,\theta,z,t)=w_0(x,\theta,t)
\end{aligned} \quad (4)$$

where u, v and w are the displacement components in the x, θ and z directions, respectively; u_0, v_0 and w_0 are the middle surface displacements of the shell in the axial, circumferential and radial directions, respectively; φ_x and φ_θ represent the transverse normal rotations of the reference surface about the θ-and x-axis. t is the time variable.

The linear strain-displacement relations in the shell space are defined as:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \gamma_{xz}\\ \gamma_{\theta z}\\ \gamma_{x\theta}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{1}{R}\left(\dfrac{\partial v_0}{\partial\theta}+w_0\right)\\ \dfrac{\partial w_0}{\partial x}+\varphi_x\\ -\dfrac{v_0}{R}+\dfrac{1}{R}\dfrac{\partial w_0}{\partial\theta}+\varphi_\theta\\ \dfrac{1}{R}\dfrac{\partial u_0}{\partial\theta}+\dfrac{\partial v_0}{\partial x}-\dfrac{v_0}{R}\end{Bmatrix}+z\begin{Bmatrix}\dfrac{\partial\varphi_x}{\partial x}\\ \dfrac{1}{R}\left(\dfrac{\partial\varphi_\theta}{\partial\theta}\right)\\ 0\\ 0\\ \dfrac{1}{R}\dfrac{\partial\varphi_x}{\partial\theta}+\dfrac{\partial\varphi_\theta}{\partial x}\end{Bmatrix} \quad (5)$$

Based on Hooke's law, the stress-strain relations of the shell are written as:

$$\begin{Bmatrix}\sigma_x\\ \sigma_\theta\\ \tau_{x\theta}\\ \tau_{xz}\\ \tau_{\theta z}\end{Bmatrix}=\begin{bmatrix}Q_{11}(z) & Q_{12}(z) & 0 & 0 & 0\\ Q_{12}(z) & Q_{11}(z) & 0 & 0 & 0\\ 0 & 0 & Q_{66}(z) & 0 & 0\\ 0 & 0 & 0 & Q_{66}(z) & 0\\ 0 & 0 & 0 & 0 & Q_{66}(z)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \gamma_{x\theta}\\ \gamma_{xz}\\ \gamma_{\theta z}\end{Bmatrix} \quad (6)$$

where $Q_{ij}(z)$ are functions of thickness coordinate z and defined as:

$$Q_{11}(z)=\frac{E(z)}{1-\mu^2(z)},\quad Q_{12}(z)=\frac{\mu(z)E(z)}{1-\mu^2(z)},\quad Q_{66}(z)=\frac{E(z)}{2[1+\mu(z)]} \quad (7)$$

The stress and moment resultants are given as:

$$(N_x,N_\theta,N_{x\theta},Q_x,Q_\theta)=\int_{-h/2}^{h/2}(\sigma_{xx},\sigma_{\theta\theta},\tau_{x\theta},\tau_{xz},\tau_{\theta z})dz \quad (8)$$

$$(M_x,M_\theta,M_{x\theta})=\int_{-h/2}^{h/2}(\sigma_{xx},\sigma_{\theta\theta},\tau_{x\theta})zdz \quad (9)$$

where N_x, N_θ and $N_{x\theta}$ are the in-place force resultants, M_x, M_θ and $M_{x\theta}$ are moment resultants, Q_x, Q_θ are transverse shear force resultants. The shear correction factor k is computed such that the strain energy due to transverse shear stresses in Eq. (9) equals the strain energy due to the true transverse stresses predicted by the three-dimensional elasticity theory [12]. In this paper k is uniformly selected by 5/6 [6]. Substituting (6)-(7) into (8)-(9), following constitutive equations are obtained:

$$\begin{Bmatrix}N_x\\ N_\theta\\ N_{x\theta}\\ M_x\\ M_\theta\\ M_{x\theta}\\ Q_x\\ Q_\theta\end{Bmatrix}=\begin{bmatrix}A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & 0 & 0\\ A_{12} & A_{11} & 0 & B_{12} & B_{11} & 0 & 0 & 0\\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0\\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & 0 & 0\\ B_{12} & B_{11} & 0 & D_{12} & D_{11} & 0 & 0 & 0\\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & k.F_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k.F_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \gamma_{x\theta}\\ k_x\\ k_\theta\\ k_{x\theta}\\ \gamma_{xz}\\ \gamma_{\theta z}\end{Bmatrix} \quad (10)$$

The materials employed in the following study are assumed to be functionally graded and linearly elastic. So, the extensional stiffness A_{ij}, the bending stiffness D_{ij}, and the extensional-bending coupling stiffness B_{ij} are respectively expressed as:

$$A_{ij}=\int_{-h/2}^{h/2}Q_{ij}(z)dz \quad B_{ij}=\int_{-h/2}^{h/2}z.Q_{ij}(z)dz$$
$$D_{ij}=\int_{-h/2}^{h/2}z^2Q_{ij}(z)dz, i,j=1,2,6 \qquad (11)$$
$$A_{ij}=\int_{-h/2}^{h/2}Q_{ij}(z)dz \quad ,i,j=4,5$$

2.3. Equations of motion

By means of Hamilton's principle, the equilibrium equations of motion based on FSDT can be written in terms of the force and moment resultants as [10]:

$$\frac{\partial N_x}{\partial x}+\frac{1}{R}\frac{\partial N_{x\theta}}{\partial\theta}=I_0\ddot{u}_0+I_1\ddot{\varphi}_x;\quad \frac{\partial N_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta}}{\partial\theta}+\frac{1}{R}Q_\theta=I_0\ddot{v}_0+I_1\ddot{\varphi}_\theta$$
$$\frac{\partial Q_x}{\partial x}+\frac{1}{R}\frac{\partial Q_{\theta}}{\partial\theta}-\frac{1}{R}N_\theta=I_0\ddot{w}_0;\quad \frac{\partial M_x}{\partial x}+\frac{1}{R}\frac{\partial M_{x\theta}}{\partial\theta}-Q_x=I_1\ddot{u}_0+I_2\ddot{\varphi}_x \qquad (12)$$
$$\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial M_{\theta}}{\partial\theta}-Q_\theta=I_1\ddot{v}_0+I_2\ddot{\varphi}_\theta$$

where $[I_0,I_1,I_2]=\int_{-h/2}^{h/2}\rho(z)\left[1,z^1,z^2\right]dz$, in which $\rho(z)$ is the density of the shell per unit middle surface area. I_0, I_1 and I_2 are the mass inertias.

3. DYNAMIC STIFFNESS MATRIX FORMULATION FOR FGM CYLINDRICAL SHELL

The state-vector is $y^T = \{u_0, v_0, w_0, \varphi_x, \varphi_\theta, N_x, N_{x\theta}, Q_x, M_x, M_{x\theta}\}^T$. Next, the Fourrier series expansion for state variables are written as:

$$\{u_o(x,\theta,t),w_o(x,\theta,t),\varphi_\theta(x,\theta,t),N_x(x,\theta,t),Q_x(x,\theta,t),M_x(x,\theta,t)\}^T$$
$$=\sum_{m=1}^{\infty}\{u_m(x),w_m(x),\varphi_{\theta m}(x),N_{x_m}(x),Q_{xm}(x),M_{x_m}(x)\}^T\cos m\theta e^{i\omega t}$$
$$\{v_o(x,\theta,t),\varphi_x(x,\theta,t),N_{x\theta}(x,\theta,t),M_{x\theta}(x,\theta,t)\}^T \qquad (13)$$
$$=\sum_{m=1}^{\infty}\{v_m(x),\varphi_{xm}(x),N_{x\theta m}(x),M_{x\theta m}(x)\}^T\sin m\theta e^{i\omega t}$$

where m is the number of circumferential wave. Substituting (13) into (10)-(11), a system of ordinary differential equations in the x-coordinate for the m^{th} mode can be expressed in the matrix form for each circumferential mode m as [6]:

$$\frac{d\mathbf{y}_m}{dx}=\mathbf{A}_m\mathbf{y}_m \qquad (14)$$

with A_m is a 10x10 matrix. The dynamic transfer matrix T_m is evaluated as:

$$T_m(\omega)=e^{\int_0^L A_m(\omega)dx}=\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix} \qquad (15)$$

Finally, the dynamic stiffness matrix $K_{(\omega)}$ for cylindrical shell is determined by:

$$K_m(\omega)=\begin{bmatrix}T_{12}^{-1}T_{11} & -T_{12}^{-1}\\ T_{21}-T_{22}T_{12}^{-1}T_{11} & T_{22}T_{12}^{-1}\end{bmatrix} \qquad (16)$$

Natural frequencies will be extracted from the harmonic responses of the structure by using the procedure detailed in [6].

4. CONTINUOUS ELEMENT FOR FGM STEPPED CYLINDRICAL SHELLS

Consider a FGM cylindrical shell with n steps as shown in Fig. 2 with following geometric parameters: R_i, L_i and h_i are radius, length and thickness of each shell step i (i = 1 - n). The cylindrical coordinate system (x, θ, z) is used for studying the whole structure.

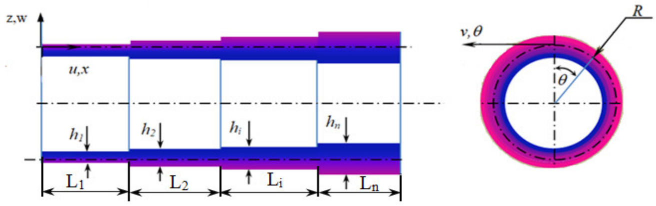

Fig. 2. Geometry of a stepped FGM cylindrical shell

In order to calculate the natural frequencies of the structure using the CEM, each shell step is represented by a single Continuous shell element. Here, it is assumed that the shell steps have the same average face and coincide with the neutral surface. Then the studied structure is constructed by an assembly of those elements.

The continuity condition will be applied to the displacements and internal forces at the neutral face of the cylindrical shell elements at the joint positions as follows:

$$N_x^{i}(1)=N_x^{i+1}(0);\ Q_x^{i}(1)=Q_x^{i+1}(0);\ N_{x\theta}^{i}(1)=N_{x\theta}^{i+1}(0);$$
$$M_{x\theta}^{i}(1)=M_{x\theta}^{i+1}(0);\ M_x^{i}(1)=M_x^{i+1}(0);\ u_i(1)=u_{i+1}(0); \qquad (17)$$
$$v_i(1)=v_{i+1}(0);\ w_i(1)=w_{i+1}(0);\ \varphi_x^{i}(1)=\varphi_x^{i+1}(0);\ \varphi_\theta^{i}(1)=\varphi_\theta^{i+1}(0)$$

where (0), (1) are the initial and final states of each cylindrical shell element.

The assembly diagram of the dynamic stiffness matrix of the FGM stepped cylindrical shell is illustrated in Fig. 3 and the equation (17) demonstrated the final DSM equation of the whole structure.

$$\begin{Bmatrix}F_1\\F_2\\ \dots\\F_{5(n+1)}\end{Bmatrix}=\begin{bmatrix}K_{1,1} & K_{1,2} & \dots & K_{1,5(n+1)}\\ K_{2,1} & K_{2,2} & \dots & K_{2,5(n+1)}\\ \dots & \dots & \dots & \dots\\ K_{5(n+1),1} & K_{5(n+1),2} & \dots & K_{5(n+1),5(n+1)}\end{bmatrix}\begin{Bmatrix}U_1\\U_2\\ \dots\\U_{5(n+1)}\end{Bmatrix} \qquad (18)$$

The natural frequencies of the structure will be obtained by solving the system of equations (18) using the harmonic response curves.

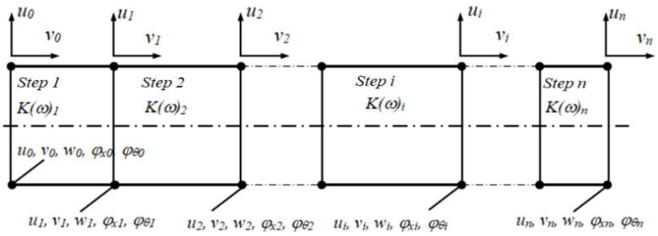

Fig. 3. The diagram of the dynamic stiffness matrix coupling of the stepped cylindrical shell

5. NUMERICAL RESULTS AND DISCUSSION

5.1. Validation of the present model

To confirm the reliability of the method, the study compared the dimensionless frequencíe of the isotropic two-step cylindrical shell ($p = \infty$) subjected to clamped-free boundary condition with the analytical results of Zhang [1] using state space techniques, Flügge shell theory and semi-analytic results of Qu et al. [3] using the segmentation method.

Table 1. Functional graded materials properties

Material / Properties	FGM1		FGM2		FGM3		FGM4		FGM5	
	Al	Si3N4	Al	Zirconia	Al	Al_2O_3	Ni	SUS304	Al	Alloy
E(Gpa)	70	322,3	70	168	70	380	205.1	207,8	70	211
μ	0.3	0,24	0.3	0.3	0.3	0.3	0.31	0,32	0.3	0.3
ρ(kg/m³)	2707	2370	2707	5700	2707	3800	8900	8166	2707	7800

All steps of the shell are made by the same property of $FGM5_{I(a=1/b=0.5/c=2/p=10)}$ material. Geometrical parameters of the structure are: $L/R = 5$; 10; $h_1/R = 0.01$; $R = 1m$; $h_2/h_1 = 0.5$; $L_1/L = 0.5$; $n = 1$, $m = 1 - 5$. The dimensionless frequencies $\Omega = \omega R(\rho(1-\mu^2)/E)^{1/2}$ of two-step metal cylindrical shells calculated by different methods are compared in Table 2.

From Table 2, it is seen that the difference between the results of CEM and those of other methods is less than 5% which is excellent. The reason for the error can be explained as follows: Zhang et al used the Flügge shell theory meanwhile our research used Reissner-Midlin's first-order shear strain shell theory. The numerical results obtained from the CEM program have high convergence and accuracy, which has confirmed the reliability of CEM.

Table 2. Dimensionless frequency comparison of two-step metal cylindrical shells

L/R	m	C-F			
		Zhang [1] A	Qu [3]	CEM B	Diffrent (%) |(A-B)*100/A|
5	1	0,097836	0,097836	0,104253	4,56
	2	0,037795	0,037807	0,038874	2,85
	3	0,022384	0,022411	0,022382	0,01
	4	0,025720	0,025746	0,025327	1,53
	5	0,036488	0,036509	0,035929	1,53
10	1	0,029463	0,029471	0,030039	1,95
	2	0,010863	0,010877	0,010602	2,40
	3	0,012904	0,012918	0,012369	4,15
	4	0,021767	0,021782	0,021204	2,59
	5	0,034280	0,034298	0,032984	3,78

5.2. Influences of shell parameters

5.2.1. The influence of the power-law exponent p on the free frequency of the FGM stepped cylindrical shell

Consider a cylindrical shell with 4 steps made of FGM5 and with clamp-clamp (C-C) boundary conditions. The steps of the shell have the same FGM material which is $FGM5_{I(a=1/b=0.5/c=2/p)}$. The SCS has the geometric parameters $R = 1m$; $h_1/R = 0.01m$; $h_2/h_1 = 2$; $h_3/h_1 = 3$; $h_4/h_1 = 4$; $L/R = 4$; $L_1/L = L_2/L = L_3/L = 1/4$.

Table 3. The effect of the power-law exponent *p* on the free frequency of the FGM 4 stepped cylindrical shell

m \ p	0	1	5	20	50	∞
1	278	277,5	274,5	273	272,5	272
2	152	151,5	150	149,5	149	148,5
3	102,5	102,5	102	102	101	100
4	106,5	107	107,5	108,5	106,5	104,5
5	134	134	135,5	137	134	131

The influence of the power-law exponent p on the free frequencies for a 4-step cylindrical shell made of FGM5I(a = 1/b = 0.5/c = 2/p = 1, 5, 20) has detailed in Table 3 and also investigated using harmonic response curves shown in Fig. 4.

From the obtaiend results in Table 3 and Fig. 4, it is noted that the shells with exponent $p = 0$ (FGM is ceramic) give the highest free vibrational frequency values because the ceramic has high elastic modulus thus the structure has the highest stiffness. On the contrary, the shells using FG material with the larger p exponent give smaller the free vibrational frequency results, and the shells with $p = \infty$ (FGM is metal) give the smallest free frequencies due to the lowest stiffness of shell structure.

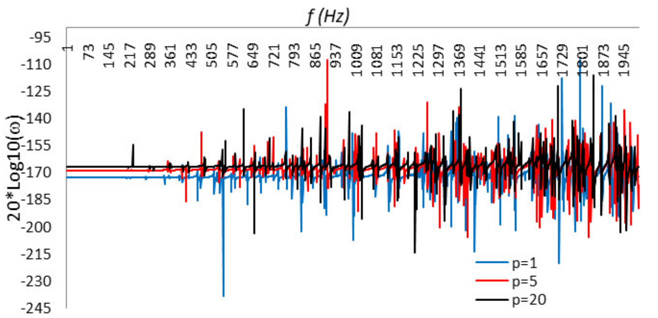

Fig. 4. Response curve of the 4 stepped cylindrical shell when the exponent p changes

5.2.2. The effect of FG material on the free frequencies of the SCS

In this section, the natural frequencies are computed for stepped cylindrical shells made of different material namely: FGM1, FGM2, FGM3, FGM4, FGM5. All shell steps have the same type of volume ratio function as FGMI(a = 1/b = 1/c = 1/p = 10), the geometrical parameters of the SCS are $R = 1m$; $h_1/R = 0.02m$; $h_2/h_1 = 1$; $L/R = 2$; $L_1/L = L_2/L = L_3/L = 1/2$. For this study, the boundary condition of the shell is F-C.

The compared results of the shell natural frequencies are described in Table 4 which shows that the stepped cylindrical shell made of FGM1 has the highest frequency and the shell made of FGM4 gives the lowest frequency. It is clearly to remark that the properties of the FGM1 (see Table 1) makes the structure have the highest stiffness. Similarly, SCS made by FGM4 gives the lowest stiffness.

Table 4. The effect of FGM on the free vibration frequency of the stepped cylindrical shell

FGM type / Frequency(Hz)	FGM1	FGM2	FGM3	FGM4	FGM5
f_1	129	70	108	61	63
f_2	146	71	124	65	67
f_3	175	96	144	84	86
f_4	230	127	190	111	114
f_5	278	136	236	124	128

6. CONCLUSIONS

In this research, a Continuous Element model for FGM stepped cylindrical shells has been successfully constructed. The influence of shell and material parameters on the free vibration of the structure has been examined. Very good agreements are noticed between the results obtained by our approach and those of other methods. From the above results, it can be concluded that:

The change of the exponent p causes the change of volume fraction of ceramic and metal according to different rules. However, when $p = 0$, the FGM is completely ceramic, so the structure has the highest stiffness resulting to the highest natural frequencies, and when $p = \infty$, the FGM is completely metal, so the structure has the lowest stiffness and the lowest frequencies are obtained.

The FGM property has a significant influence on the natural frequencies of the stepped cylindrical shell. FGM of entire ceramic has high elastic modulus and small density which gives the structure an important rigidity, thus the calculation results of free frequencies are high.

The developed CE model with its powerful assembling procedure can be expanded to study more complex shell structures such as: joined cylindrical-conical shells, combined cylindrical-conical shell and annular plates, ring-stiffened shells and those structures surrounded by elastic foundations.

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